Derivation of the Random Walk Model
Computational Cognitive Science 2010

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Assume that the learner currently has a set of n perceptual observations that are available to guide the decision-making. In Lecture 8 we denoted these observations $\mathbf{v}_A$ and $\mathbf{v}_B$, to indicate that these correspond to actual sensory "samples" of the stimuli A and B. However, for the sake of simplicity, I'll let $\mathbf{v} = (\mathbf{v}_A, \mathbf{x}_B)$ denote the complete set of data; similarly $v_i = (v_{Ai}, v_{B_i})$ from the original notation. Along much the same lines, I'll truncate the notation associated with the decision: instead of writing $A > B$, I'll just write A, indicating that the decision-maker chooses option A.

Bayes' theorem states that the probability that the correct answer is A can be given as:

$$P(A|\mathbf{v}) = \frac{P(\mathbf{v}|A)P(A)}{P(\mathbf{v})}. \tag{1}$$

Therefore, the posterior odds ratio comparing the relative plausibility of A and B is given by

$$\frac{P(A|\mathbf{v})}{P(B|\mathbf{v})} = \frac{P(\mathbf{v}|A)}{P(\mathbf{v}|B)} \times \frac{P(A)}{P(B)} \tag{2}$$

If we assume that the v_i values are approximately conditionally independent, then this becomes

$$\frac{P(A|\mathbf{v})}{P(B|\mathbf{v})} \approx \prod_{i=1}^{n} \frac{P(v_i|A)}{P(v_i|B)} \times \frac{P(A)}{P(B)} \tag{3}$$

Taking logarithms gives us:

$$\ln \frac{P(A|\mathbf{v})}{P(B|\mathbf{v})} \approx \sum_{i=1}^{n} \ln \frac{P(v_i|A)}{P(v_i|B)} + \ln \frac{P(A)}{P(B)}. \tag{4}$$

We now define the log-odds on the LHS to be x_n,

$$x_n = \ln \frac{P(A|\mathbf{v})}{P(B|\mathbf{v})} \tag{5}$$

and each of the log-odds terms on the RHS to be one of the y_i values:

$$y_i \quad = \quad \ln \frac{P(v_i|A)}{P(v_i|B)} \tag{6}$$

$$y_0 = \ln \frac{P(A)}{P(B)} \tag{7}$$

If we substitute these new variables back into Eq. 4, then we get the additive expression:

$$x_n = \sum_{i=0}^{n} y_i \tag{8}$$

At this point, it helps to recall that we are thinking about observations that arrive over time. So it is convenient to switch subscripts now, using t instead of n.

$$\begin{aligned} x_t &= \sum_{i=0}^{t} y_i && (9) \\ &= y_t + \sum_{i=0}^{t-1} y_i && (10) \\ &= y_t + x_{t-1} && (11) \end{aligned}$$

Thus, when data arrive over time, we can characterise the Bayesian updating process via this very simple difference equation. Now, recall from the slides in Lecture 7, we stated that our goal was to continue sampling until the probability of making an error is sufficiently low. If this error tolerance is ϵ, we either want it to be the case that $P(A|\mathbf{v}) < \epsilon$ or $P(B|\mathbf{v}) < \epsilon$. Since this is a two-choice problem, $P(B|\mathbf{v}) = 1 - P(A|\mathbf{v})$, so we can restate our goal as follows. We continue to sample, as long as

$$\epsilon < P(A|\mathbf{v}) < 1 - \epsilon \tag{12}$$

At this stage, it would be nice to figure out what this means for x_t. Note that:

$$\begin{aligned} x_t &= \ln \frac{P(A|\mathbf{v})}{P(B|\mathbf{v})} && (13) \\ \exp(x_t) &= \frac{P(A|\mathbf{v})}{1 - P(A|\mathbf{v})} && (14) \\ \exp(-x_t) &= \frac{1 - P(A|\mathbf{v})}{P(A|\mathbf{v})} && (15) \\ 1 + \exp(-x_t) &= \frac{1}{P(A|\mathbf{v})} && (16) \\ \frac{1}{1 + \exp(-x_t)} &= P(A|\mathbf{v}) && (17) \end{aligned}$$

so we can rewrite Eq 12 as

$$\epsilon < \frac{1}{1 + \exp(-x_t)} < 1 - \epsilon \tag{18}$$

Now, we can apply the reverse trick:

$$
\begin{aligned}
\frac{1}{1+\exp(-x_t)} &> \epsilon & (19)\\
1+\exp(-x_t) &< \frac{1}{\epsilon} & (20)\\
\exp(-x_t) &< \frac{1-\epsilon}{\epsilon} & (21)\\
\exp(x_t) &> \frac{\epsilon}{1-\epsilon} & (22)\\
x_t &> \ln\frac{\epsilon}{1-\epsilon} & (23)
\end{aligned}
$$

We now define

$$\gamma = \ln\frac{\epsilon}{1-\epsilon} \tag{24}$$

which means that we can now restate Eq. 18 as follows: continue to sample until

$$|x_t| < \gamma \tag{25}$$

where $|\cdot|$ denotes the absolute value function. Thus, the sampling algorithm is

```
set t = 0
set x_0 to reflect prior beliefs
while |x_t| < γ
  time increments:  t = t + 1
  draw new observation:  v_t
  calculate the associated log-odds:  y_t
  update beliefs:  x_t = x_{t-1} + y_t
make decision (at time t):
  if x_t ≥ γ, choose A
  if x_t ≤ −γ, choose B
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